AN APPLICATION OF DYNAMIC PROGRAMMING
TO THE DETERMINATION OF
OPTIMAL SATELLITE TRAJECTORIES

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SUMMARY

We consider a simplified satellite trajectory problem, corresponding to a flat earth assumption, first treated by Okhotsimskii and Eneev. We present a numerical solution based upon the functional equation technique of dynamic programming, and a proof of the fundamental result in the analytic solution.

The same computational approach can be applied to more realistic trajectory problems.

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1. Introduction

In this paper we wish to consider the problem of determining optimal satellite trajectories. To illustrate the general techniques of dynamic programming, we shall consider in detail a simplified satellite trajectory problem, posed and treated by D. E. Okhotsimskii and T. M. Eneev in a paper originally published in Russian in [1]*, and then published in English translation in [2]**.

Despite superficial evidence of the correctness of their theories and results, it would appear that the mathematical argumentation in their papers is at best incomplete. One purpose of this paper is to deduce one of thei· principal results rigorously using standard variational arguments. What is rather interesting is that we arrive at a variational problem of unconventional type which has not been treated to any extent in the literature.

The principal part of our paper is devoted to the computational solution of the problems by means of the functional equation technique of dynamic programming. We shall present

*[1]. Uspekhi Fizicheskikh Nauk, September, 1957.

** [2]. J. British Interplanetary Soc., Jan.-Feb., 1958.

some numerical results. The importance of this approach lies in the fact that we can solve in the same manner variational problems which defy precise analysis.

2. The Simplified Problem

We wish to ascertain the thrust control policy and fuel consumption regime which will put a satellite into orbit at a specified altitude with maximum horizontal component of velocity.

Essential simplifications arise from the neglect of aerodynamic forces and the assumption that the terrestrial gravitational field is plane-parallel.

Determination of paths of minimum fuel, maximum altitude, and so on, can be treated along the same lines as the following discussion.

3. Mathematical Formulation

The equations of motion of a satellite traveling over a flat earth in a Cartesian coordinate system will be taken to be

$$\text{(1)} \qquad \frac{du}{dt} = p \cos \phi$$

$$\frac{dw}{dt} = p \sin \phi - g$$

$$\frac{dy}{dt} = w$$

$$\frac{dx}{dt} = u.$$

Here

(2) (a) x and y are, as usual, the horizontal and vertical coordinates,

(b) u and w are the horizontal and vertical projections of velocity,

(c) p is the magnitude of acceleration due to reaction force,

(d) ϕ is the inclination of the thrust to the horizontal.

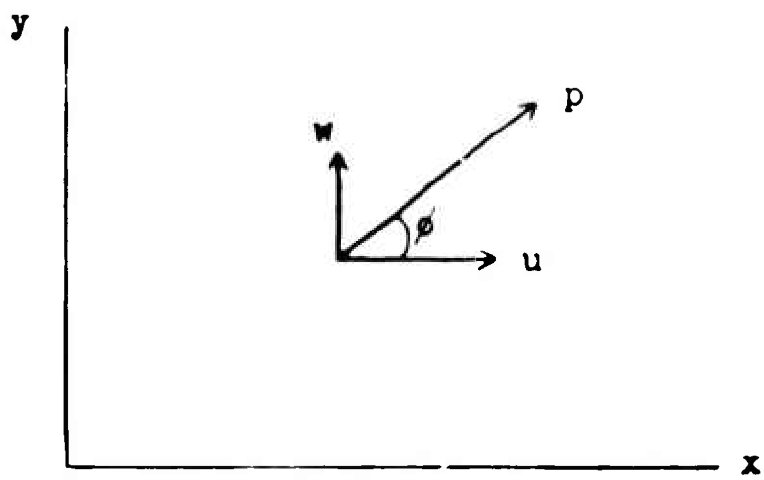


Fig. 1

If we introduce the quantity V as the velocity available to the satellite in the idealized case of no gravitational force, we obtain the relation

(3) $$\frac{dV}{dt} = -p.$$

The variable V will be a monotone function of the quantity of fuel. Since

(4) $$p = \frac{gP}{M},$$

where P is the thrust, and

(5) $$P = -\frac{c}{g}\frac{dM}{dt},$$

where M is the weight and c is the exit velocity of the gases, we can solve for M in terms of the "ideal available velocity," V, obtaining the equation

(6) $$M = M_e e^{V/c},$$

where M_e is the weight of the empty rocket.

The equations of motion, (3.1), together with (6) preceding, which yields mass as a function of V, and (4) above, giving acceleration in terms of thrust and mass, enable us to determine optimal inclinations and optimal magnitude of thrust as functions of V. In the next section we shall consider the associated variational problem.

4. Variational Formulation

A direct statement of the problem of determining optimal thrust and inclination leads to immediate difficulties because of the linearity of the equations and criterion function. An optimal policy would consist of either zero or infinite accelerations. Since this last is physically meaningless, in order to pose a sensible variational problem we impose a constraint of maximum

possible thrust, Q. Furthermore, in these initial sections we shall agree to burn all the fuel at maximum allowable rate and then coast.

This assumption will not be made in later sections where we obtain a direct computational solution using dynamic programming techniques. As will be seen, even under these simplifying assumptions, we are led to variational problems of some novelty and interest.

With the foregoing assumptions, the problem posed verbally in §2 becomes that of determining the inclination function $\phi(V)$ which maximizes

$$(1) \qquad J(\phi) = -\int_{V_0}^{0} \cos \phi \, dV,$$

subject to the restriction that

$$(2) \qquad \int_{V_0}^{0} \left(\frac{dy}{dV} + \frac{w}{g}\frac{dw}{dV}\right) dV = H,$$

and the relations in (3.1) and (3.3) - (3.6). Here

(3) (a) H is the prescribed altitude,

(b) V_0 is the initial ideal velocity.

The functional in (2) evaluates the altitude gained during burning together with the altitude obtained after burnout due to the vertical component of velocity at burnout.

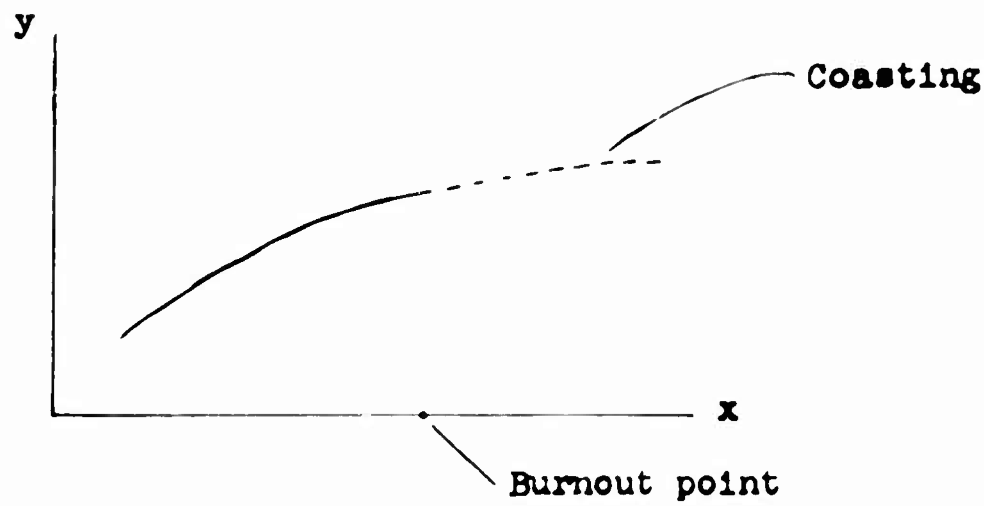


Fig. 2

5. Preliminary Transformation

Referring to the equations of (3.1), we see that the constraint in (4.2) may be written in the form

(1) $$\int_{V_0}^{0} \frac{w \sin \phi}{g}\, dV = -H.$$

Turning back to (3.1), we may write

(2) $$w(V) = w_0 - \int_{V_0}^{V} (\sin \phi - \frac{g}{p}) dV.$$

Using this relation in (5.1), we obtain the relation

(3) $$\int_{V_0}^{0} \sin \phi\, r(V) dV + \frac{1}{2g} \left(\int_{V_0}^{0} \sin \phi dV \right)^2 = H,$$

where the function $r(V)$ is given by

$$(4) \qquad r(V) = \frac{w_0}{g} - \int_{V_0}^{V} \frac{dV}{p} = \frac{w_0}{g} - \frac{cM_e}{gP}\left(e^{V/c} - e^{V_0/c}\right)$$

$$= k_1 + k_2 e^{V/c},$$

where k_1 and k_2 are constants.

6. Discussion

We thus obtain the problem of maximizing $J(\phi)$, as given by (4.1), subject to the constraint of (3) above. This is representative of an interesting class of variational problems which do not appear to have been discussed in any detail heretofore.

The general problem would be that of maximizing a functional

$$(1) \qquad J_1(y) = \int_0^T F_1(x,y)dt,$$

subject to constraints of the form

$$(2) \qquad J_2(y) = G\left(\int_0^T F_2(x,y)dt, \int_0^T F_3(x,y)dt, \ldots, \int_0^T F_N(x,y)dt\right) = k.$$

We shall present a formal analysis, postponing a more rigorous discussion until a later date.

7. Variational Analysis

As usual, we set

(1) $\qquad \phi = \bar{\phi} + \epsilon\Psi,$

where ϵ is an infinitesimal. We have

(2) $\qquad J(\phi) = J(\bar{\phi}) - \epsilon \int_{V_0}^{0} \Psi \sin \bar{\phi} \, dV,$

and (5.3) yields

(3) $$\int_{V_0}^{0} \Psi \cos \bar{\phi} \, r(V) dV + \frac{1}{g} \left(\int_{V_0}^{0} \sin \bar{\phi} \, dV \right) \left(\int_{V_0}^{0} \Psi \cos \bar{\phi} \, dV \right) = 0.$$

Since Ψ is arbitrary, it follows that there exists a constant λ such that

(4) $$\sin \bar{\phi} = \lambda \left[r(V) \cos \bar{\phi} + \frac{1}{g} \left[\int_{V_0}^{0} \sin \bar{\phi} \, dV \right] \cos \bar{\phi} \right].$$

Since $\int_{V_0}^{0} \sin \bar{\phi} \, dV$ is a constant, albeit unknown, and

(5) $$\frac{d}{dt} r(V) = k_2 \frac{d}{dt} e^{V/c} = \frac{k_2}{c} e^{V/c} \frac{dV}{dt}$$

$$= \frac{k_2}{c} e^{V/c} \frac{gP}{M_e} e^{-V/c} = \frac{gP}{M_e c} \cdot \frac{M_e c}{gP} = 1,$$

it follows that we obtain the important conclusion that the optimal policy, $\bar{\phi}$, is characterized by the property that

(6) $\qquad \frac{d}{dt} \tan \bar{\phi} = \lambda,$

<u>a constant</u>.

This agrees with the result claimed in [2].

8. Dynamic Programming Approach—I

Let us now see how we can employ the functional equation approach of dynamic programming to obtain a computational solution. The basic idea is to regard the problems of the calculus of variations as particular examples of multi-stage decision processes of continuous type. This approach is discussed in some detail in [3] ***, and applied to optimal trajectory problems in [4] ****.

The state variables are altitude y, vertical component of velocity w, and ideal available velocity V. Consequently, we introduce the function

(1) $f(V,w,y)$ = the additional horizontal velocity obtained starting at altitude y, vertical component of velocity w and ideal available velocity V, and using an optimal policy.

Referring to the defining equations of motion in §3, and using the Principle of Optimality as in [3] or [4], we obtain the functional equation

*** [3]. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

**** [4]. Cartaino, T., and S. Dreyfus, "Application of dynamic programming to the airplane minimum time-to-climb problem," Aeronautical Engineering Review, vol. 16, no. 6, 1957, pp. 74-77.

$$(2)\qquad f(V,w,y) = \underset{\phi,P}{\mathrm{Max}}\left[\cos\phi\Delta V + f\left(V-\Delta V, w + \frac{dw}{dV}\Delta V, y + \frac{dy}{dV}\Delta V\right)\right]$$

$$= \underset{\phi,P}{\mathrm{Max}}\left[\cos\phi\Delta V + f\left(V-\Delta V, w + \left(\sin\phi - \frac{M_e e^{V/c}}{P}\right)\Delta V,\; y + \frac{wM_e e^{V/c}}{gP}\Delta V\right)\right]$$

where ΔV is regarded as a small quantity.

Letting V assume only a finite set of values $0, \Delta V, 2\Delta V, \ldots, N\Delta V$, we see that the computation becomes that of determining a sequence of functions of two variables $f_N(w,y) \equiv f(N\Delta, w, y)$, using (2).

9. Dynamic Programming Approach—II

In order to simplify the computation, we use a Lagrange multiplier formalism, as discussed in [5]*****, to reduce the problem to one of determining a sequence of functions of one variable.

In place of maximizing $J(\phi)$ subject to the constraint of (5.1), we consider the problem of maximizing

$$(1)\qquad -\int_{V_0}^{0} \cos\phi\, dV + \lambda\int_{V_0}^{0} \frac{dy}{dV}\, dV,$$

subject to the constraints of the equations of motion. Here λ is the Lagrange parameter.

*****[5]. Bellman, R., "Dynamic programming and Lagrange multipliers," Proc. Nat. Acad. Sci. USA, vol. 42, 1956, pp. 767-769.

The new functional equation for the maximum value is

$$(2) \qquad f(V,w) = \operatorname*{Max}\left\{ \begin{array}{l} \operatorname*{Max}_{\phi,P}\left[\cos\phi\Delta V + \dfrac{\lambda w M_e e^{V/c}}{gP}\Delta V\right. \\ \qquad \left. + f\left(V - \Delta V, w + \left(\sin\phi - \dfrac{M_e e^{V/c}}{P}\right)\Delta V\right)\right], \\ \lambda w\Delta t + f(V, w - g\Delta t) \end{array} \right\}$$

where the second alternative within the Max $\{\ \}$ represents a decision to coast for a small time interval Δt.

The parameter λ is adjusted until the altitude constraint of (4.2) is met. By using the Lagrange parameter, we have partitioned a problem originally involving a sequence of functions of two variables into a set of problems involving functions of one variable. The gain in computing time and effort is considerable.

10. Computational Aspects

The numerical solution is obtained by iterating the recurrence equation (9.2) backwards from the known final values. The calculation is begun by observing that, if burnout occurs with a vertical component of velocity w, the additional altitude obtained during coasting will be $w^2/2g$ and the additional horizontal velocity will be zero. Hence $f(0,w) = \lambda w^2/2g$. A table containing $f(0,w)$ for a range of w values (we do not yet know to what burnout value of w the optimal policy will lead) is stored in the high speed memory of the computer.

This tabular function is now used to determine a new function, $f(\Delta V,w)$, the total additional horizontal velocity plus λ times the altitude that can be attained starting with a small quantity ΔV of "available velocity (fuel)" and vertical velocity component w. This calculation is performed using equation (4.2). We actually evaluate the gain associated with choices of different ϕ's and P's and compare this with the return from a decision to coast. On this basis we pick the optimal decision. The return from this decision is recorded in the computer memory as the value of $f(\Delta V,w)$ for the particular value of w considered. A second table is constructed giving the optimal decision that yielded $f(\Delta V,w)$. A third table, $J(\Delta V,w)$, is maintained giving the total altitude gained when following an optimal path starting from $(\Delta V,w)$. Since we are flying so as to maximize horizontal velocity plus λ times altitude, this third table is just a convenient record that is not used in the calculation, but which, when the iteration of equation (4.2) is finished, yields immediately the total altitude (and hence the horizontal velocity, $f(V,0) - \lambda J(V,0)$), gained by following an optimal trajectory.

Once the technique described above for calculating $f(\Delta V,w)$ using the table of $f(0,w)$ has been programmed for a computer, it is a simple matter to have the same code calculate $f(2\Delta V,w)$ from $f(\Delta V,w)$ and, finally, $f(V,w)$ from $f(V - \Delta V,w)$. Notice that at each stage of this computation only one table of the function f is needed to

compute the next table in the sequence. Once a table has been computed and used in the calculation of the next table it can be printed by the computer and destroyed in memory. Hence the computer memory capacity required is determined by the number of discrete points chosen for the w-table, and does not depend on the fineness of the ΔV grid. The total time for a calculation depends inversely on the size of ΔV.

At the completion of the backwards iteration of equation (4.2) one knows the horizontal velocity and altitude obtained by an optimal policy for the specified initial conditions. Also the initial decision for the starting point is determined by the nature of the calculation of $f(V,w)$. To reconstruct the optimal path in its entirety one now determines the new value of w after using the prescribed decision for the first ΔV interval. In calculating $f(V - \Delta V,w)$ for this w-value, an optimal decision was determined and recorded (since the actual w may not be a point of the w-grid, interpolation may be necessary) and this decision is used during the interval $V - \Delta V$ to $V - 2\Delta V$. In this manner we use the output of the sequence of calculations, processing them in the opposite order from that in which they were computed.

The above operation may be performed easily by the computer as the final step of the calculation if the requisite tables are stored on tape or punched into cards.

Once the problem has been solved, one examines the final altitude to determine if the required height was attained. A

new value of the Lagrange multiplier λ is then calculated based upon previous values and results and the calculation is repeated.

One calculation yields the optimal path, in terms of horizontal velocity vs. altitude, for a wide range of initial vertical components of velocity. This variety of results is of interest in problems where initial vertical velocity is not necessarily specified and the answers for a range of values is desired. Secondly, after several variations of λ, optimal trajectories to several different altitudes are known, yielding an interesting estimate of the trade-off between altitude and velocity along optimal trajectories.

11. Numerical Results

For all calculations, we have assumed a hypothetical missile with the following characteristics:

> **Empty weight, M_e = 5000 lbs.**
>
> Exhaust velocity, c = 11,000 ft./sec.
>
> Maximum thrust, P_{max} = 300,000 lbs.
>
> Minimum thrust with engine on, P_{min} = 50,000 lbs.
>
> Total ideal available velocity = 30,000 ft./sec.

These data imply a total weight at takeoff of 76,456 lbs.

A value of λ of .00142 yielded a final altitude of approximately 450 miles with a horizontal component of

velocity at this altitude of 26,300 ft./sec.

Various parameters and grid-sizes required for numerical solution were chosen as follows:

1. $\Delta V = 1000$. Therefore the recurrence relation was iterated 30 times.

2. $\Delta w = 50$. Each table of $f(V,w)$ contained 281 numbers, since w was allowed to assume value from 0 to 14,000.

3. $\Delta \phi = .01$ radian. Admissible thrust angles were $0, .01, .02, \ldots, \pi/2$ radians.

4. Thrust could assume values 300,000, 250,000, 200,000, 150,000, 100,000, or 50,000.

These numbers were determined experimentally. They possess the property that a further refinement has little or no effect on the computed solution.

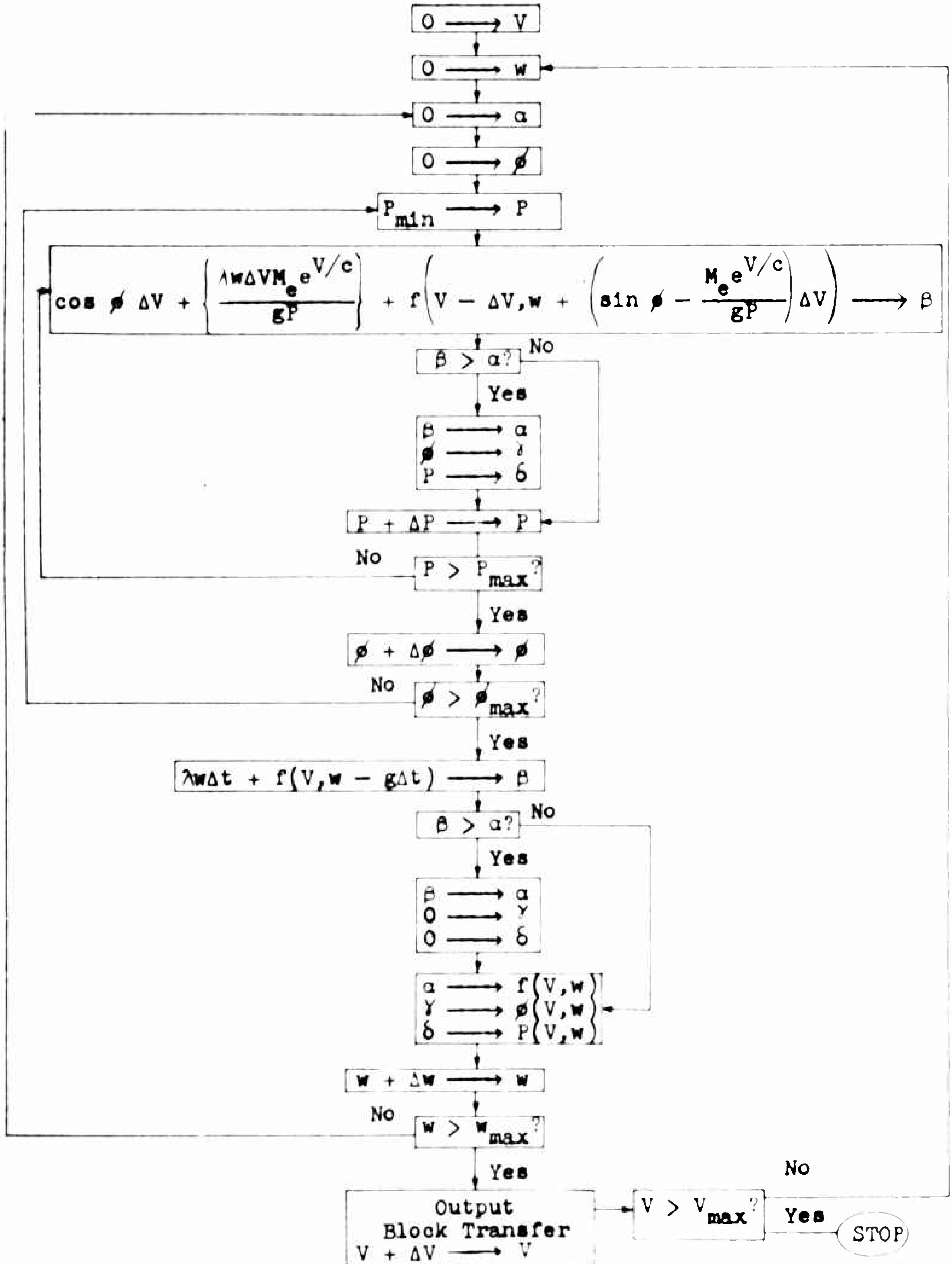
A condensed summary of the solution, as computed on the RAND JOHNNIAC computer in 20 minutes, is shown below.

It should be noted that, although in this simplified study the rocket is flown at maximum thrust until burnout and the thrust direction obeys a simple law, the computational scheme assumes neither of these results. It is therefore applicable to more general problems that are not amenable to conventional mathematical analysis.

V (ft/sec)	Mass (lb)	h (ft)	w (ft/sec)	u (ft/sec)	ϕ (rad.)	P (lbs)	Time (sec)
30,000	76,456	0	0	0	.560	300,000	0
25,000	48,529	16,045	1,514	4,279	.523	300,000	33.3
20,000	30,803	57,720	3,306	8,625	.501	300,000	54.4
15,000	19,552	106,344	5,259	13,020	.490	300,000	67.8
10,000	12,410	152,954	7,330	17,436	.480	300,000	76.3
5,000	7,877	192,848	9,464	21,871	.480	300,000	81.7
burnout	5,000	224,920	11,650	26,313	.472	300,000	85.1
end of coast	5,000	2,337,679	0	26,313	0	0	444.7

12. Flow-chart

A flow diagram of the program is shown below:


0 → V
0 → w
0 → α
0 → ø
P_min → P
cos ø ΔV + {λwΔVM_e e^{V/c} / gP} + f(V − ΔV, w + (sin ø − M_e e^{V/c} / gP) ΔV) → β
β > α?
No
Yes
β → α
ø → γ
P → δ
P + ΔP → P
No
P > P_max?
Yes
ø + Δø → ø
No
ø > ø_max?
Yes
λwΔt + f(V, w − gΔt) → β
β > α?
No
Yes
β → α
0 → γ
0 → δ
α → f(V,w)
γ → ø(V,w)
δ → P(V,w)
w + Δw → w
No
w > w_max?
Yes
Output
Block Transfer
V + ΔV → V
V > V_max?
No
Yes
STOP